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Abstract. This paper studies a modified of dynamics of Leslie-Gower predator-prey population model. The model is stated as a system of first order differential equations. The model consists of one predator and one prey. The Holling type II as a predation function is considered in this model. The predator and prey populations are assumed to be beneficial and then the two populations are harvested with constant efforts. Existence and stability of the interior equilibrium point are analysed. Linearization method is used to get the linearized model and the eigenvalue is used to justify the stability of the interior equilibrium point. From the analyses, we show that under a certain condition the interior equilibrium point exists and is locally asymptotically stable. For the model with constant efforts of harvesting, cost function, revenue function, and profit function are considered. The stable interior equilibrium point is then related to the maximum profit problem as well as net present value of revenues problem. We show that there exists a certain value of the efforts that maximizes the profit function and net present value of revenues while the interior equilibrium point remains stable. This means that the populations can live in coexistence for a long time and also maximize the benefit even though the populations are harvested with constant efforts.

1. Introduction

Mathematical modeling associated with the dynamics of predator and prey populations has become a great research theme in the field of mathematical ecology and in the field of fisheries management. There are some fenomena in the fisheries includes population as a predator and the other population as a prey. The dynamics of the predator and prey populations does not include only two populations, but some fenomena show the involvement of more than two populations in the environment. Since the population, for example, fish gives benefit then the population is considered as a stock. The population is then managed to give more benefits and the population remains sustainable. There are some prey-predator models with harvesting. The predator-prey fishery model with selective harvesting for prey, see [1-4], selective harvesting for predator, see [5- 8], and the two populations are harvested, see [9].

The dynamics of predator and prey populations via Lotka-Volterra model have been extensively considered by many authors. In Lotka-Volterra model, the predator just depends on the size of prey population. This model does not consider the situation when the size of prey decreases then the predator will seek other prey. Leslie modeled the effect of this fenomena. Leslie-Gower predator-prey model is another approach to the dynamics of predator-prey. This model is structurally different from the classical Lotka-Volterra model. Some researchers have considered and modified the Leslie-Gower model with various aspects, see [10-12]. Most of predator-prey fisheries models with harvesting are related to the economic problems including maximum profit and total discounted net revenue, see [1, 6, 9].

Based on the works above, we present a modified Leslie-Gower predator-prey model with constant efforts of harvesting for the two populations. We analyse the stability of the interior equilibrium point and not only to determine the critical value of the efforts that maximize the profit but also to maintain the stability of the equilibrium point. The optimal harvesting policy of present value of revenue is solved using Pontryagin's maximal principle.

2. Leslie-Gower predator-prey model

We consider a predator-prey fishery in an environment which includes two populations, predator and prey. If we let $x(t)$ denotes the size of prey and let $y(t)$ be the size of predator at time t, then the growth rate of predator-prey populations based on the Leslie-Gower model is given by following system

$$\begin{aligned}\frac{dx}{dt} &= (r - bx)\,x - p(x)\,y\\ \frac{dy}{dt} &= \left(s - \frac{ay}{x}\right)y\,.\end{aligned} \tag{1}$$

The function $p(x)$ in model (1) is a predation functional response which measures the level of predator consumption to the prey. Firstly, Leslie considered the function $p(x)$ is proportional to the size of prey. The meaning of all parameters in model (1) can be seen, for example, in [6, 11]. By considering the Holling type II functional response of the predator to the prey, the model is then modified and becomes

$$\begin{aligned}\frac{dx}{dt} &= \left(r - bx - \frac{a_1 y}{k_1 + x}\right)x\\ \frac{dy}{dt} &= \left(s - \frac{a_2 y}{k_2 + x}\right)y\,.\end{aligned} \tag{2}$$

We consider that predator and prey populations are economically valuable, thus these two populations are harvested with constant efforts. Model (2) is then extended and becomes

$$\begin{aligned}\frac{dx}{dt} &= \left(r - bx - \frac{a_1 y}{k_1 + x}\right)x - q_1 E_1 x\\ \frac{dy}{dt} &= \left(s - \frac{a_2 y}{k_2 + x}\right)y - q_2 E_2 y\,.\end{aligned} \tag{3}$$

In the model (3), parameters q_1 and q_2 state the cathability coefficient for the prey and the predator populations respectively. Parameters E_1 and E_2 state the harvesting efforts satisfying the conditions $0 \le E_i \le E_{i\max}$ for $i = 1, 2$ and some values of $E_{i\max}$. For the analysis, we let $r_1 = r - q_1 E_1$ and $s_1 = s - q_2 E_2$. From the model (3), we have five non negative equilibrium points, namely $T_0 = (0,0)$, $T_1 = \left(0, \frac{s_1 k_2}{a_2}\right)$, $T_2 = \left(\frac{r_1}{b}, 0\right)$, $T_3 = (x_3, y_3)$, and $T_4 = (x_4, y_4)$, where $x_3 = \frac{-A_2 + \left(A_2^2 - 4A_1A_3\right)^{1/2}}{2A_1}$, $y_3 = \frac{s_1(x_3 + k_2)}{a_2}$, $x_4 = \frac{-A_2 - \left(A_2^2 - 4A_1A_3\right)^{1/2}}{2A_1}$, $y_4 = \frac{s_1(x_4 + k_2)}{a_2}$,

$A_1 = ba_2 > 0$, $A_2 = s_1a_1 - r_1a_2 + k_1ba_2$, and $A_3 = s_1k_2a_1 - r_1k_1a_2$. In order for the equilibrium points T_3 and T_4 be exist in the first quadrant we assume that $A_2^2 - 4A_1A_3 \geq 0$.

Case 1. $A_3 = s_1k_2a_1 - r_1k_1a_2 < 0$, the equilibrium point T_3 becomes an interior equilibrium point while the equilibrium point T_4 will not be considered because $x_4 < 0$.

Case 2. $A_2 = s_1a_1 - r_1a_2 + k_1ba_2 < 0$ and $0 < A_3 < \frac{A_2^2}{4A_1}$, the equilibrium points T_3 and T_4 become interior equilibrium point. In this case, the equilibrium point T_3 may be stable or not stable while the equilibrium point T_4 is an unstable saddle point.

For analysis we use the Jacobian matrix to get the linear model and by evaluating the Jacobian matrix at the equilibrium point $T_3 = (x_3, y_3)$, we have

$$J_T = \begin{pmatrix} r_1 - 2bx_3 + \frac{a_1y_3x_3}{(k_1+x_3)^2} - \frac{a_1y_3}{(k_1+x_3)} & -\frac{a_1x_3}{(k_1+x_3)} \\ \frac{a_2{y_3}^2}{(k_2+x_3)^2} & s_1 - \frac{2a_2y_3}{(k_2+x_3)^2} \end{pmatrix}. \quad (4)$$

The polynomial characteristic (4) associated with the Jacobian matrix J_T is given by $f(\lambda) = \det(\lambda I - J_T)$, i.e $f(\lambda) = \lambda^2 - (d_1 + d_4)\lambda + (d_1d_4 - d_2d_3)$, where

$d_1 = r_1 - 2bx_3 + \frac{a_1y_3x_3}{(k_1+x_3)^2} - \frac{a_1y_3}{(k_1+x_3)}$, $d_2 = -\frac{a_1x_3}{(k_1+x_3)} < 0$, $d_3 = \frac{a_2{y_3}^2}{(k_2+x_3)^2} > 0$, and $d_4 = s_1 - \frac{2a_2y_3}{(k_2+x_3)^2}$. Referring to the Routh-Hurwitz criterion [13], the interior equilibrium point T_3 is locally asymptotically stable when the conditions $d_1 + d_4 < 0$ and $d_1d_4 - d_2d_3 > 0$ are satisfied.

Example 1. For model (2), we set the parameter values as $r = 12$, $b = 0.001$, $s = 9$, $a_1 = 2.5$, $a_2 = 2.1$, $k_1 = 5$, and $k_2 = 7$ in appropriate units. Then we have equilibrium points $T_0 = (0,0)$, $T_1 = (0,30)$, $T_2 = (1200,0)$, $T_3 = (1268.89, 5468.11)$, and $T_4 = (11.82, 80.66)$. For the equilibrium point T_3, we get eigenvalues from the Jacobian matrix as $0.2100 \pm 3.3503i$ which means the equilibrium point T_3 is unstable spiral. For the equilibrium point T_4, we get the eigenvalues from the associated Jacobian matrix as 0.5414 and -3.1284 which means that the equilibrium point T_4 is unstable saddle point, see figure 1. It is easy to check from the phase portrait that the positive x axis and positive y axis are stable manifold while the equilibrium point T_4 is unstable saddle point and equilibrium point T_3 is unstable spiral. In this situation, there exist a stable limit cycle in the first quadrant.

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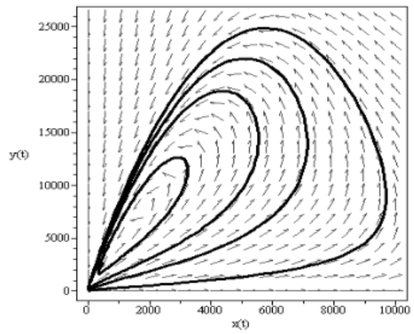

Figure 1. Plot of trajectories around the interior equilibrium points.

Example 2. For model (2), we set the parameter values as $r = 12$, $b = 0.001$, $s = 8$, $a_1 = 1$, $a_2 = 3$, $k_1 = 10$, and $k_2 = 5$ in appropriate units. The equilibrium points of the model are $T_0 = (0,0)$, $T_1 = (0,13.33)$, $T_2 = (12000,0)$, $T_3 = (9334.76, 24906.02)$, and $T_4 = (-11.42, -17.13)$. In this case, the equilibrium point T_4 does not exist in the first quadrant. The equilibrium point T_3 is asymptotically stable with the eigenvalues $-7.3362 \pm 4.5683i$, see figure 2.

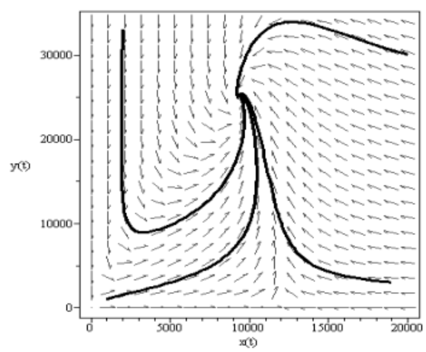

Figure 2. Plot of trajectories around the equilibrium point T_3

3. Bionomic equilibrium

The biological equilibrium is found by solving the system $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$. The economic equilibrium is obtained when the proceeds from the sale of harvested biomass equals to the costs used in harvesting activities. We assume that $TR = pY(E)$, where $Y(E) = EqN$ is the yield of harvesting, p is the price of the biomass, E is harvesting effort, q is the cathability coefficient, and N is biomass. We also assume that total cost of harvesting is proportional the harvesting effort, $TC = cE$. The profit function is defined as $\pi = TR - TC$. The profit function of harvesting for the modified Leslie-Gower predator-prey model is $\pi(E_1, E_2) = (p_1 q_1 x)E_1 + (p_2 q_2 y)E_2 - (c_1 E_1 + c_2 E_2)$. The bionomic equilibrium (x^*, y^*, E_1^*, E_2^*), see [14], is found by solving simultaneously the equations $\left(r - bx - \frac{a_1 y}{k_1 + x}\right)x - q_1 E_1 x = 0$, $\left(s - \frac{a_2 y}{k_2 + x}\right)y - q_2 E_2 y = 0$, and $\pi(E_1, E_2) = 0$.

We then relate the stable interior equilibrium point T_3 to the maximum profit problem. The equilibrium point T_3 is an interior equilibrium if the conditions $0 \le E_i \le E_{i\max}$ for $i = 1, 2$ and $A_3 = s_1k_2a_1 - r_1k_1a_2 < 0$, that is

$$q_1k_1a_2E_1 - q_2k_2a_1E_2 < rk_1a_2 - sk_2a_1, \tag{5}$$

are satisfied. The profit function at the equilibrium point T_3 is written in the form

$$\pi(E_1, E_2) = (p_1q_1x)E_1 + (p_2q_2y)E_1 - (c_1E_1 + c_2E_2).$$

The problem is determining a pair of efforts (E_1, E_2) which satisfies condition (5) and $0 \le E_i \le E_{i\max}$ for $i = 1, 2$ that maximizes the profit function $\pi(E_1, E_2)$. We also need to maintain the interior equilibrium point T_3 is always asymptotically stable.

Example 3. For the problem of maximum profit function, we set the parameter values as $r = 12$, $b = 0.001$, $s = 8$, $a_1 = 1$, $a_2 = 3$, $k_1 = 12$, $k_2 = 5$, $q_1 = 1$, and $q_2 = 1$ in appropriate units. Take $p_1 = 10$, $p_2 = 15$, $c_1 = 5$, and $c_2 = 5$ in appropriate units. Then we have the equilibrium point $T_3 = (x_3, y_3)$, where

$$\begin{aligned} x_3 &= 4661.6667 + 166.6667E_2 - 500E_1 \\ &\quad + 166.6667\left[(-27.970 - E_2 + 3E_1)^2 + 3.840 - 0.360E_1 + 0.060E_2\right]^{1/2}, \\ y_3 &= (2.6667 + 0.3333E_2)\Big\{4666.6667 + 166.6667E_2 - 500E_1 \\ &\quad + 166.6667\left[(-27.970 - E_2 + 3E_1)^2 + 3.840 - 0.360E_1 + 0.060E_2\right]^{1/2}\Big\}. \end{aligned}$$

In order for equilibrium point T_3 becomes an interior point, the harvesting efforts E_1 and E_2 must satisfy the conditions $A_3 < 0$ where $A_3 = 30E_1 - 5E_2 - 320$ and $0 \le E_i(t) \le E_{i\max}$ for $i = 1, 2$. We put $E_{1\max} = 5$ and $E_{2\max} = 5$. In the other words, the equilibrium point T_3 becomes an interior point when $(E_1, E_2) \in D_1$, where $D_1 = \{(E_1, E_2): 0 \le E_1 \le 5, 0 \le E_2 \le 5, 30E_1 - 5E_2 < 320\}$. The profit function associated with the equilibrium point T_3 is given by

$$\pi(E_1, E_2) = (p_1q_1x_3)E_1 + (p_2q_2y_3)E_2 - (c_1E_1 + c_2E_2).$$

After substituting the values of x_3 and y_3, and then simplifying we get

$$\begin{aligned} \pi(E_1, E_2) = &\Big\{46611.6667 + 1666.6667E_2 + 5000E_1 \\ &+ 1666.6667\left[(-27.970 - E_2 + 3E_1)^2 + 3.840 - 0.360E_1 + 0.060E_2\right]^{1/2}\Big\}E_1 \\ &+ \Big\{(40 - 5E_2)\Big[4666.6667 + 166.6667E_2 - 500E_1 \\ &+ 1666.6667\left((-27.970 - E_2 + 3E_1)^2 + 3.840 - 0.360E_1 + 0.060E_2\right)^{1/2}\Big] - 5\Big\}E_2. \end{aligned}$$

We have a stationary point $(E_1^*, E_2^*) = (1.41693, 4.33906)$ in D_1. The only critical point $(E_1^*, E_2^*) = (1.41693, 4.33906)$ satisfies the conditions and also maximizes the profit with the value of $\pi(E_1^*, E_2^*) = 876733.5362$. By applying the value of harvesting efforts

$\left(E_1^*, E_2^*\right)=(1.41693, 4.33906)$ we get the equilibrium point $T_3=(9361.3614, 11449.1077)$. The polynomial characteristic of the Jacobian matrix at the equilibrium point T_3 is given by $f(\lambda)=\lambda^2+11.80804\lambda+34.33137$ which has the eigenvalues -6.62936 and -5.17869. Under this situation, with the values of efforts at the level of $E_1^*=1.41693$ and $E_2^*=4.33906$, then the predator and the prey populations will sustain for a long period of time and also maximize the profit function.

4. Optimal harvesting policy

Our objective in this problem is to maximize the net present value of revenues which is given by

$$J=\int_0^\infty e^{-\delta t}\left[(p_1q_1x-c_1)E_1(t)+(p_2q_2y-c_2)E_2(t)\right]dt. \tag{6}$$

The symbol δ states the instantaneous rate of discount. We need to maximize J subject to the constraint equation (3) by using Pontryagin's maximum principle [15]. The control variables $E_1(t)$ and $E_2(t)$ are subject to the constraints $0\le E_i(t)\le E_{i\max}$, for $i=1,2$.

The Hamiltonian equation is

$$H=e^{-\delta t}\left[(p_1q_1x-c_1)E_1+(p_2q_2y-c_2)E_2\right]+\lambda_1\left(rx-bx^2-\frac{a_1xy}{k_1+x}-q_1E_1x\right)+\lambda_2\left(sy-\frac{a_2y^2}{k_2+x}-q_2E_2y\right). \tag{7}$$

We set $\frac{\partial H}{\partial E_1}=0$ and $\frac{\partial H}{\partial E_2}=0$ as the necessary conditions for the control variables E_1 and E_2 to be optimal. From the Hamiltonian function (7), we have $\frac{\partial H}{\partial E_1}=e^{-\delta t}(p_1q_1x-c_1)-\lambda_1q_1x=0$ and $\frac{\partial H}{\partial E_2}=e^{-\delta t}(p_2q_2y-c_2)-\lambda_2q_2y=0$. Then we get $\lambda_1=\frac{e^{-\delta t}(p_1q_1x-c_1)}{q_1x}$ and $\lambda_2=\frac{e^{-\delta t}(p_2q_2y-c_2)}{q_2y}$.

From the Hamiltonian equation we also have

$$\frac{\partial H}{\partial x}=e^{-\delta t}p_1q_1E_1+\lambda_1\left[r-2bx+\frac{a_1xy}{(k_1+x)^2}-\frac{a_1y}{(k_1+x)}-q_1E_1\right]+\frac{\lambda_2a_2y^2}{(k_2+x)^2} \text{ and}$$

$$\frac{\partial H}{\partial y}=e^{-\delta t}p_2q_2E_2-\frac{\lambda_1a_1x}{(k_1+x)}+\lambda_2\left[s-\frac{2a_2y}{(k_2+x)}-q_2E_2\right].$$

From the Pontryagin's maximum principle $\dot{\lambda}_1=-\frac{\partial H}{\partial x}$, $\dot{\lambda}_2=-\frac{\partial H}{\partial y}$, and we get

$$\frac{\delta e^{\delta t}(-p_1q_1x+c_1)}{q_1x}+e^{-\delta t}p_1q_1E_1+\lambda_1\left[r-2bx+\frac{a_1xy}{(k_1+x)^2}-\frac{a_1y}{(k_1+x)}-q_1E_1\right]+\frac{\lambda_2a_2y^2}{(k_2+x)^2}=0, \tag{8}$$

$$\frac{\delta e^{-\delta t}(-p_2q_2y+c_2)}{q_2y}+e^{-\delta t}p_2q_2E_2-\frac{\lambda_1a_1x}{(k_1+x)}+\lambda_2\left[s-\frac{2a_2y}{(k_2+x)}-q_2E_2\right]=0. \tag{9}$$

By substituting $\lambda_1=\frac{e^{-\delta t}(p_1q_1x-c_1)}{q_1x}$ and $\lambda_2=\frac{e^{-\delta t}(p_2q_2y-c_2)}{q_2y}$ into the equations (8) and (9) we get $E_1=E_1(x,y)$ and $E_2=E_2(x,y)$. Again, after substituting $x=x_3=\frac{q_3E_3+k}{a_5}$ and

$y = y_3 = \frac{s_2 + \left(s_2{}^2 + 4s_1 a_1 x_3\right)^{1/2}}{2s_1}$ into the equations $E_1 = E_1(x, y)$ and $E_2 = E_2(x, y)$ we get the value of control variables E_1 and E_2. The values of E_1, E_2, x_3, and y_3 maximize the present value of revenues *J*.

Example 4. For the problem of maximizing present value of the net revenues, we set the parameter values as $r = 12$, $b = 0.001$, $s = 8$, $a_1 = 1$, $a_2 = 3$, $k_1 = 10$, $k_2 = 5$, $q_1 = 1$, and $q_2 = 0.9$ in appropriate units. Take $p_1 = 15$, $p_2 = 20$, $c_1 = 10$, $c_2 = 15$, and $\delta = 0.005$ in appropriate units. Then we have the equilibrium point $T_3 = (x_3, y_3)$, where

$$x_3 = 4661.6667 + 150.0000E_2 - 500E_1 + 166.6667\left[(-27.970 - 0.9E_2 + 3E_1)^2 + 3.840 - 0.360E_1 + 0.0540E_2\right]^{1/2},$$

$$y_3 = (2.6667 + 0.3000E_2)\left\{4666.6667 + 150.0000E_2 - 500E_1 + 166.6667\left[(-27.970 - 0.9E_2 + 3E_1)^2 + 3.840 - 0.360E_1 + 0.0540E_2\right]^{1/2}\right\}.$$

The adjoint variables are $\lambda_1 = \frac{5e^{-0.005t}(3x_3 - 2)}{x_3}$ and $\lambda_2 = \frac{3.3333e^{-0.005t}(6y_3 - 5)}{y_3}$. We get the efforts $E_1 = 2.63403$ and $E_2 = 8.80361$, $T_3 = (9340.39754, 239.09227)$. The eigenvalues relates to the equilibrium are -9.31464 and -0.07696. The adjoint variables are $\lambda_1 = 14.99893e^{-0.005t}$ and $\lambda_2 = 19.93029e^{-0.005t}$. Therefore we get the maximum value of present value of the net revenues $J = \int_0^{\infty} 4.0677297 \cdot 10^5 \; e^{-0.005t} dt = 8.135459 \cdot 10^7$.

5. Conclusions

The modified Leslie-Gower predator-prey population model with constant efforts of harvesting may has two interior equilibrium points, namely $T_3 = (x_3, y_3)$ and $T_4 = (x_4, y_4)$. When the equilibrium point T_3 exists in the first quadrant, it may be stable or unstable. When the equilibrium T_4 exists in the first quadrant, it is always unstable saddle point. When the equilibrium points T_3 and T_4 exist in the first quadrant where the equilibrium point T_3 is unstable spiral, there exists a stable limit cycle in the first quadrant.

With the restriction of harvesting efforts, $0 \le E_i \le E_{i\,\max}$, there exists a certain condition such that the interior equilibrium point T_3 remains stable and also gives maximum profit. The predator and the prey populations can live in coexistence, although the two populations are harvested with constant efforts. By using Pontryagin's maximum principle, we found that there exists a certain value of harvesting efforts, E_1 and E_2, which is associated with the stable equilibrium point T_3 that maximize the net present value of revenues.

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